12.0 What Do We Mean by "Elementary Geometry"?

p. 381

The main topic of this chapter is Euclidean geometry. Based on definitions and axioms described in Euclid's *Elements*, **Euclidean geometry** is sometimes referred to as *elementary geometry*. This chapter is also elementary, or basic, in the sense that the mathematical function concept is not required, and a coordinate system is not used.

12.1 Geometric Elements and Figures

We will study concepts and names of geometric elements and figures, a sometimes dreary pursuit but giving the necessary preparation for more gratifying study, such as geometric construction (Section 12.3) and theorems (Section 12.4).

Points

A geometric **point** has no dimension – is void of quantity – and therefore cannot be drawn as "just a point". Thus, the concept of a geometric point is **axiomatic**.

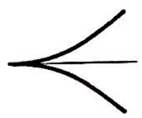
x
o

A point is often represented by a small cross – signifying the intersection of two lines – or a small circlet.

Points associated with plane curves often have special names; see, *e.g.*, the definition of *tangent*, on page 388.

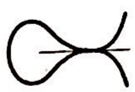
Cusp

A **cusp** is a double point on a curve where the curve has two coincident tangents.



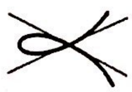
A **cusp of the first kind** is a point where the curve has a branch on each side of the common tangent near the point of contact.

A **cusp of the second kind** is a point where the two parts of the curve lie on the same side of the common tangent near the point of contact.



A double cusp – or **point of osculation** – is a point where the two branches of the curve have a common tangent, each branch extending in both directions of the tangent.

Node, Crunode



A **node**, or **crunode**, is a point where two branches of a curve *cross* and have different tangents.

Salient point

At a **salient point** two branches of the curve *meet and stop*, and have different tangents.

Locus

A **locus** (plural: loci) is a geometric figure for which all of its points satisfy a given condition. For instance, the locus of all points in a plane at equal distance from a given point is a circle; the locus of all points in space with the same distance from a given point is a sphere.

Lines

A moving point describes a **line** that has only length, but no breadth. As with the point, the concept of a line is **axiomatic.**

We find ourselves with the slightly paradoxical truth that

two nonparallel lines define a point

and

two points define a line;

Dual Elements

Dual Operations

in this sense points and lines are said to be **dual elements**, and the intersection of lines to give points and the connection of points to give lines are **dual operations.**

A **straight line** – usually called just a **line** – is the shortest distance between two points. It may extend without limit, or be a **determinate straight line** – also called a **line segment** – which contains both of the endpoints and all points between them; the length of the line segment is the **distance** between the endpoints.

A succession of straight line segments is a **broken line.**

Two or more lines in the same plane are **parallel lines** if they do not meet however far they extend.

Transversal

A line that intersects two or more (often parallel) lines is a **transversal**.

Perpendicular

Latin, *per*, "through" *pendere*, "to weigh"

A line that meets another line at right angles is a **perpendicular**, indicated by a small square in the angle.

Crossing Line

Skew Line

Lines which lie in different planes and do not intersect each other are **crossing lines** or **skew lines**.

Ray (Half-Line)

Origin

A straight line issuing from an **initial point**, often denoted O for **origin**, is a **ray** or a **half-line**; a **closed ray** includes the origin, an **open ray** does not.

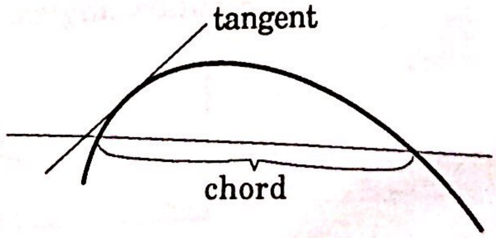
Curve, Arc

A **curve** is any continuous image of a line or segment; an **arc** is a portion of a curve, as in an *arc of a circle*. Curves are usually curved, but they can have straight sections.

Secant

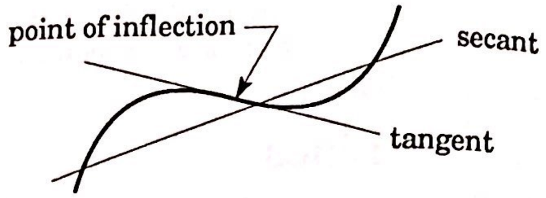
Chord

A straight line that intersects a curve is called a **secant**; the part of the secant contained between the points of intersection is a **chord**.



Tangent

A straight line that just touches a curve – a limit position of the secant – is a **tangent**; it has a double point of contact with the curve.



Inflection Tangent

A tangent to a curve in a point of inflection – where the curve changes its direction of curvature – is an **inflection tangent**; it has three points in common with the curve (these notions are best understood in the context of *calculus*, which will be discussed in later chapters).

Perpendicular
Normal

A straight line at right angles to the tangent in its point of contact with the curve is a **perpendicular** to the tangent and a **normal** to the curve.

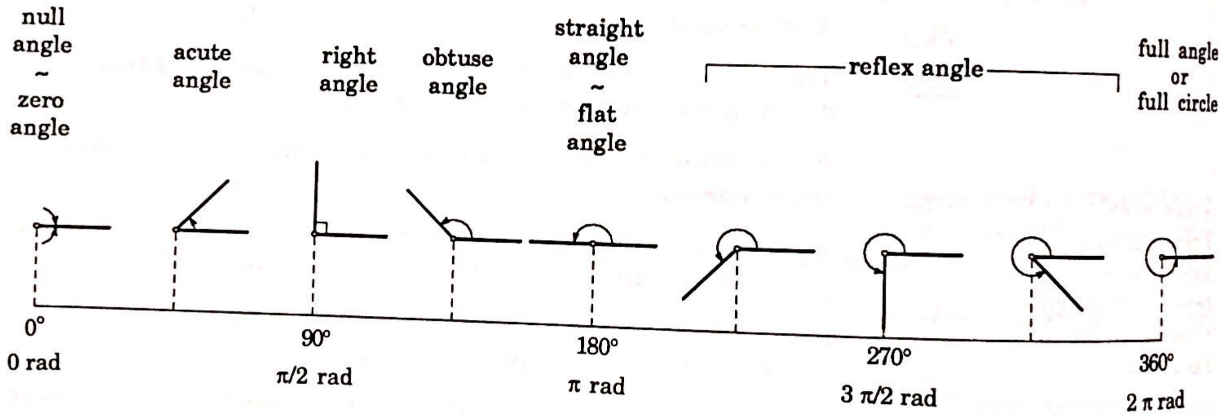
Angles

∠

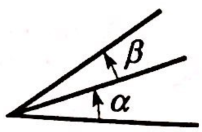
Latin, *vortex*, "whirl", "summit", "top of the head"

A **plane angle** (∠), or simply an **angle**, is formed by two rays – sides or legs of the angle – which extend from a common point, the **vertex** of the angle.

Angles have different names:

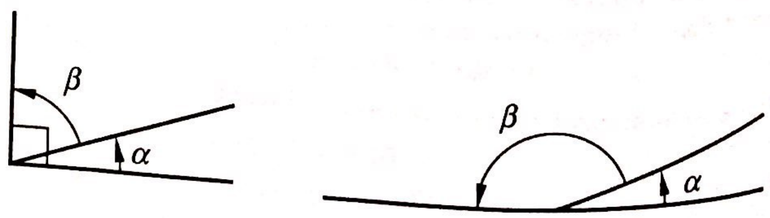


Two angles with a leg in common are **adjacent angles**.



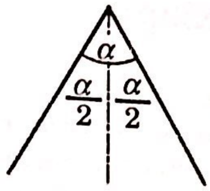
Complementary Angles
Supplementary Angles

Two angles whose sum is a right angle are **complementary angles**; two angles whose sum is a straight angle are **supplementary angles**:



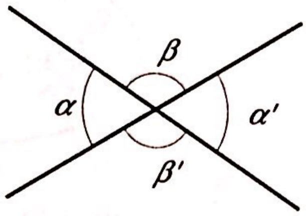
Bisector

A **bisector** is a straight line which divides an angle into two equal angles.



Vertical Angles

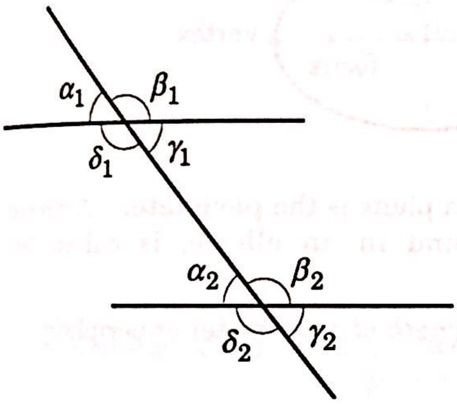
Two intersecting straight lines form two pairs of **vertical angles**; vertical angles are equal.



vertical angles $\begin{cases} \alpha = \alpha' \\ \beta = \beta' \end{cases}$

These angles are called *vertical* because each side of one is an extension through the *vertex* of a side of the other.

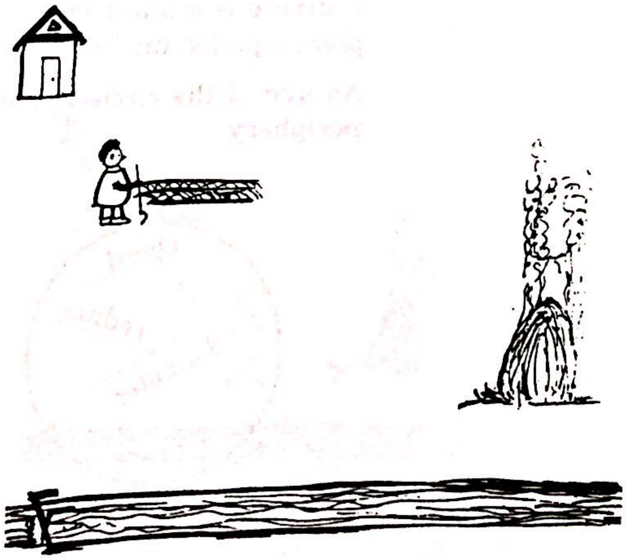
A transversal which intersects a pair of parallel lines produces many pairs of angles, some of which are equal, as shown below:



$\alpha_1 \quad \beta_1 \quad \gamma_2 \quad \delta_2$	are exterior angles
$\alpha_2 \quad \beta_2 \quad \gamma_1 \quad \delta_1$	are interior angles
$\gamma_1 = \alpha_2;\ \delta_1 = \beta_2$	are alternate interior angles
$\alpha_1 = \gamma_2;\ \beta_1 = \delta_2$	are alternate exterior angles
$\alpha_1 = \alpha_2;\ \beta_1 = \beta_2$ $\gamma_1 = \gamma_2;\ \delta_1 = \delta_2$	are corresponding angles
$\alpha_1 = \gamma_1;\ \beta_1 = \delta_1$ $\alpha_2 = \gamma_2;\ \beta_2 = \delta_2$	are opposite angles

* * *

Joe was out hoeing beans when he noticed smoke from the hay field. He grabbed two buckets from the shed and, via the edge of the irrigation ditch, took the shortest way to the burning hay. Living and working by the maxim "The shortest distance between two points is a straight line", how did Joe set his course?

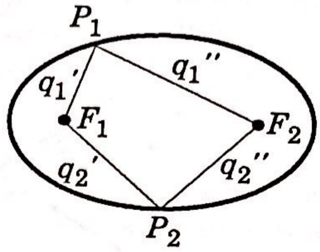


[Answer on page 391]

Ellipses and Circles

Ellipses

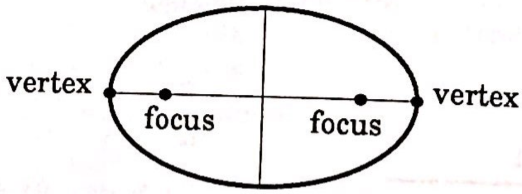
An ellipse is a plane curve where the sum of the distances between two fixed points (the **foci**) and any point on the periphery is constant; below, F_1 and F_2 are fixed points and P_1 and P_2 are arbitrary points on the periphery:



$$q_1' + q_1'' = q_2' + q_2''$$

Major Axis
Minor Axis

An ellipse has two axes of symmetry, a **major axis**, between the **vertices** of the ellipse, and a **minor axis**, which intersect at the center of the ellipse:



Perimeter

Periphery

The boundary of a figure in a plane is the **perimeter**. A curved perimeter, such as is found in an ellipse, is called the **periphery**.

Circumference

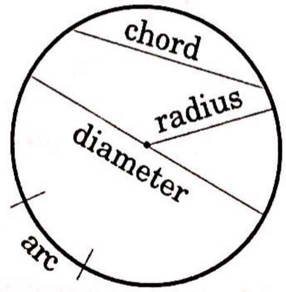
The **circumference** is the *length* of a perimeter or periphery.

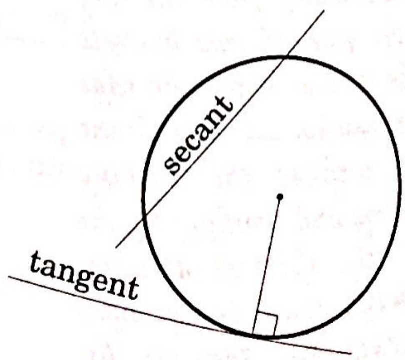
Circles

A **circle** is a plane curve that is the locus of all points in the plane equidistant from a given point, the **center** of the circle.

Arc
Periphery

An **arc** of the circle is bounded by two distinct points on the **periphery**.





Secant
Chord
Diameter

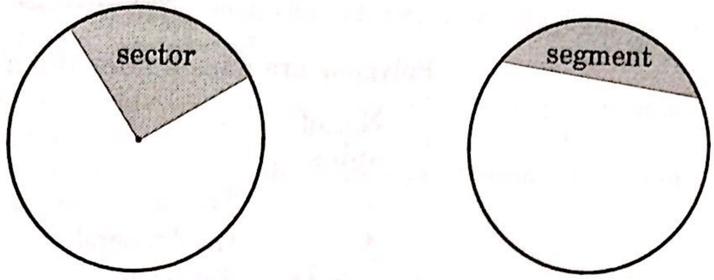
A **secant** of a circle is a straight line intersecting the periphery in two points; a **chord** is the part of the secant within the circle. A chord passing through the center is the **diameter**– the longest chord of the circle.

Tangent
Radius

A **tangent** is a straight line which has a double point of contact with the periphery, and may be regarded as the limit position of a secant. The tangent is perpendicular to the **radius** of the circle at the point of contact.

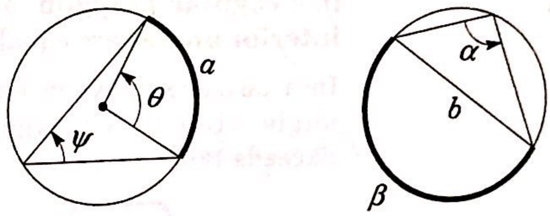
Sector
Segment
Subtend: to be opposite to and mark off

A **sector** of the circle is bounded by two radii and the included arc; a **segment** is bounded by a chord and the arc subtending the chord:



Angle Subtended by an Arc

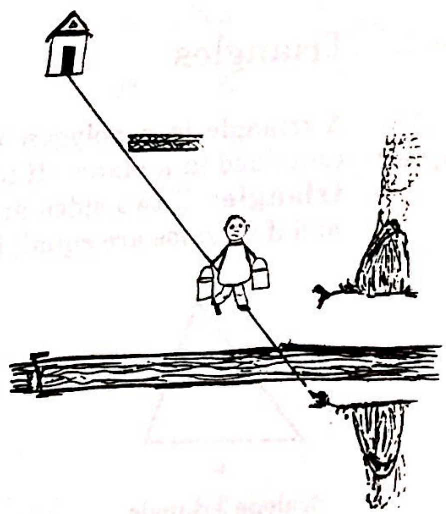
A **central angle**, θ, **subtended by an arc**, a, as shown below, is an angle whose vertex is at the center and whose sides are radii. The **peripheral angle**, ψ, subtended by the same arc a, has half the measure of θ:



Angle Subtended by a Chord

The **angle α subtended by a chord** b has its vertex on the periphery and its two sides are chords that together with b form a triangle as shown below. Because α is the peripheral angle for the arc β, its measure is independent of where its vertex falls in the complement of β.

* * *



Answer to the haystack problem from page 389: To hit the point on the edge of the irrigation ditch that gives the shortest total distance to the haystack, Joe envisioned a mirror image of the haystack (the edge of the ditch closest to Joe being the "mirror") and a straight line between the shed and the mirror image.

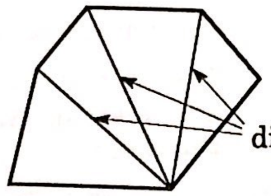
Polygons

Greek, *póly*, "many"; *gonía*, "angle", from *góny*, "knee"

A **polygon** is a plane figure with three or more angles and as many sides. It is bounded by a broken line forming a simple closed curve, a circuit without self-intersections.

Diagonal

A **diagonal** is a straight line connecting two opposite vertices.



Polygons are named after the number of sides or vertices:

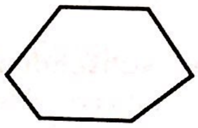
No. of sides	Name	No. of sides	Name
3	Triangle	9	Nonagon
4	Quadrilateral Tetragon	10	Decagon
5	Pentagon	11	Undecagon
6	Hexagon	12	Dodecagon
7	Heptagon		
8	Octagon	n	n-gon

Regular Polygon

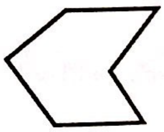
In a **regular polygon**, all sides have the same length, and all **interior angles** are equal.

Convex
Concave

In a **convex** polygon each interior angle is less than a flat angle (180°); a polygon is **concave** if an interior angle exceeds 180°:



Convex Hexagon



Concave Hexagon

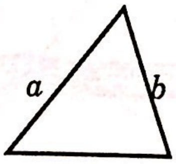
Reentrant Angle
Salient Angle

The inward-pointing angle of the concave polygon is a **re-entrant angle**; the other angles are **salient angles**.

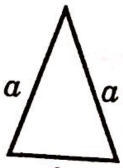
Triangles

Greek, *skalinós*, "uneven"
Greek, *iso-*, "equal", *skélos*, "leg"

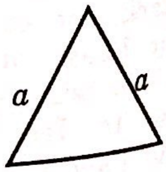
A triangle is a polygon with three sides. Every triangle is contained in a plane. If no two sides are equal, it is a **scalene triangle**; if two sides are equal, it is an **isosceles triangle**; and if all sides are equal, it is an **equilateral triangle**.



Scalene Triangle

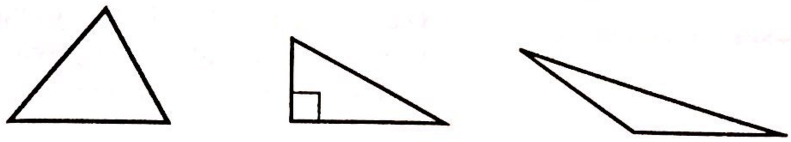


Isosceles Triangle



Equilateral Triangle

If each interior angle of a triangle is less than a right angle, we speak of an **acute triangle**; if one angle is greater than a right angle, we have an **obtuse triangle**. Acute and obtuse triangles are called **oblique triangles**. A triangle with a right angle is called a right-angled triangle or a **right triangle**.

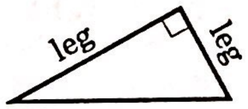


Egyptian Triangle

A right triangle with the sides of 3, 4, and 5 units is known as an **Egyptian triangle**.

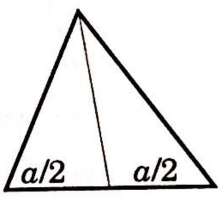
Leg Base
Hypotenuse
Greek, *hypoteínoysa*, "stretching under"

The two equal sides of an isosceles triangle are known as the **legs**, the third side being the **base**. In a right triangle, the side opposite the right angle is the **hypotenuse**, the other two sides being the legs.



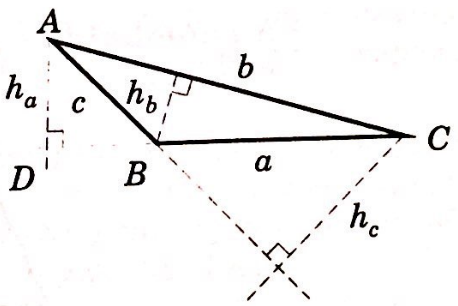
Median

A **median** of a triangle is a straight line from a vertex to the midpoint of the opposite side:



Height, Altitude
Normal, Perpendicular

The **height**, or **altitude**, of a triangle is a **normal**, or **perpendicular**, from a vertex to the opposite side or to its extension. In the triangle ABC below, h_a is the height from vertex A to the extension of side a; h_b is the height from vertex B to side b; and h_c is the height from vertex C to the extension of side c:



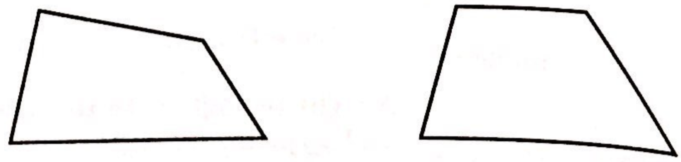
Quadrilaterals

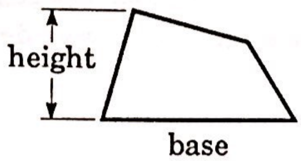
Quadrangle
Tetragon

A **quadrilateral** is a polygon with four sides; other names, less common, are **quadrangle** and **tetragon**.

Trapezoid, Trapezium

A quadrilateral with no two sides parallel is known as a **trapezoid** in the U.K., a **trapezium** (plural: trapezia) in the U.S.; if two sides are parallel, it is a trapezium in the U.K. and a trapezoid in the U.S.



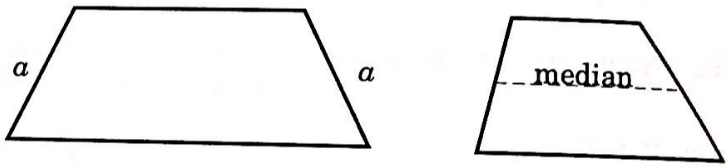


The **base** of a trapezium or trapezoid is generally its longest side, on which it is supposed to stand. If two sides are parallel, they are usually both called bases.

The **height**, or **altitude**, of a trapezium or trapezoid is the perpendicular from the highest point toward the base, or the distance between the two parallel bases.

Leg
Isosceles Trapezoid

The two nonparallel sides are called **legs**. If the legs are of equal length, we have an **isosceles** trapezium or trapezoid:



Median

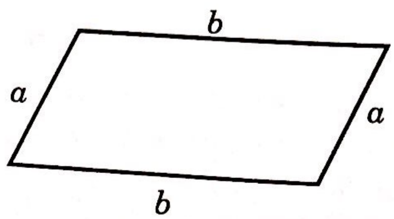
A **median**, or midline, is a straight line segment joining the midpoints of the nonparallel sides.

Parallelograms

A general parallelogram is a quadrilateral in which both pairs of opposite sides are parallel.

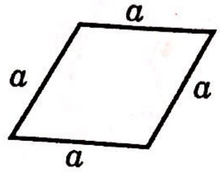
Rhomboid

A **rhomboid** is a parallelogram whose adjacent sides are unequal:



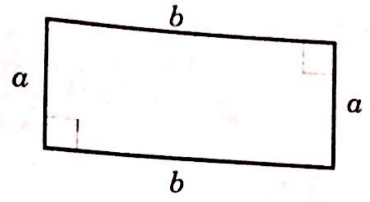
Rhombus

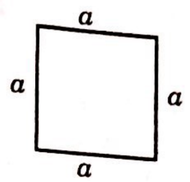
A rhomboid with all sides equal is a **rhombus** (plural: rhombi):

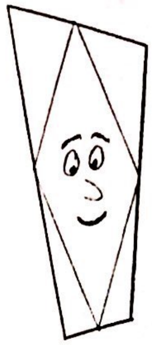


Rectangle
Square

A **rectangle** is a right-angled parallelogram; a **square** is a rectangle with all sides equal – a regular quadrilateral:







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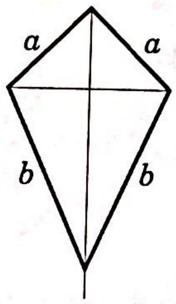
How would you construct a quadrilateral that, after joining the midpoints of its adjacent sides, includes a parallelogram?

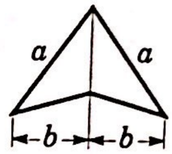
Answer: *Any quadrilateral will do. Try, and look for yourself!*

Kites and Deltoids

Kites and **deltoids** are quadrilaterals whose adjacent sides are equal in pairs.



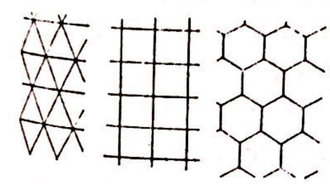
kite



deltoid

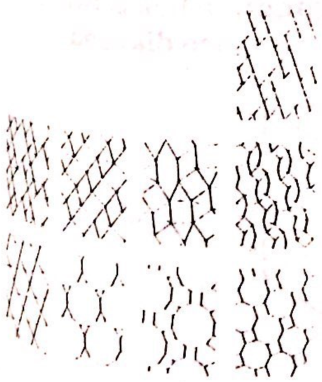
Kites and deltoids have mirror symmetry, as do rhombi.

Tiling



The mathematical study of tiling (tessellation) is concerned with how shapes can be placed to completely fill a plane or space.

The only regular polygons that – one kind at a time – can tile the plane are equilateral triangles, squares, and hexagons (*cf.* honeycombs and the crystal pattern of snowflakes, both originally investigated by Kepler).



By allowing two or more types of regular polygons to meet at each vertex but requiring the vertex configurations to be the same, you allow an additional nine tilings; two of these are mirror images of each other. Allowing more general shapes leads to a wealth of possibilities.

Tiling – in the plane and in space – forms a basis of active and challenging mathematical research with applications in graphics, networks, the study of chemical structure, and geology.